SOME REMARKS ON TRIGONOMETRIC INTERPOLATION

BY

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ABSTRACT

The object of this paper is to consider the problem of (0, 1, 2, 4) trigonometric interpolation when nodes are taken to be $x_{kn} = (2k\pi/n)$, $k = 0, 1 \ldots, n-1$. Here the interpolatory polynomials are explicitly constructed and the corresponding convergence theorem is proved, which is shown to be best possible in a certain sense. It is interesting to compare these results with those of Saxena [6], where the convergence theorem requires the existence of $f'''(x)$.

Motivated by a series of papers of P. Turán and his associates [1, 2, 7] and O. Kis [4], A. Sharma and the author [8] have recently considered the problem of existence, uniqueness, explicit representation and the problem of convergence in the $(0, M)$ case, that is, when

$$R_n(x_{kn}) = \alpha_{kn},\ R_n^{(M)}(x_{kn}) = \beta_{kn},\ x_{kn} = \frac{2k\pi}{n}, \tag{1.1}$$

$k = 0, 1, \cdots, n-1$, α_{kn}, β_{kn} are prescribed, M being a fixed positive integer $\geqq 1$. When $M = 1$, the problem has been considered by Jackson [3] and when $M = 2$, the case has been treated by O. Kis [4]. As was pointed out in our paper that the situation is different when M is even from that when M is odd. We proved that when M is even the conditions for uniform convergence require the Zygmund condition on $f(x)$, while we require $f(x)$ to be only continuous and periodic when M is odd. In other words when M is odd we provided a new proof of Weierstrass approximation theorem where the trigonometric polynomial has also the property of interpolation. The object in this paper is to generalize the results of O. Kis in a different direction viz to $(0, 1, 2, 2M)$ interpolation $(M \geqq 2)$, when $x_{kn} = (2k\pi/n)$. In order to keep the proof simpler we will discuss only $(0, 1, 2, 4)$

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case. Using the terminology used by P. Turán by (0, 1, 2, 4) interpolation we mean finding interpolatory trigonometric polynomial of order $2n$ of the form (3.2) where the value, first, second and fourth derivatives are prescribed at n distinct points $x_{kn}=(2k\pi/n)$, $k=0,1 \cdots$, $n-1$. It turns out that the trigonometric polynomial exist uniquely if n is odd. Moreover the convergence theorem in this case turns out to be very interesting. We remark that the problem of (0, 1, 2,4) interpolation on the nodes of $\pi_n(x) = (1 - x^2)P_{n-1}(x)$, $P_n(x)$ being Legendre polynomial has been considered by Saxena [6] where the convergence theorem requires $f(x)$ to be twice differentiable. Further the explicit representation of interpolatory polynomials is not simple. On the contrary in our case of (0, 1, 2, 4) interpolation we will require $f(x)$ to belong to Zygmund condition λ and it is best possible in a certain sense. Also the explicit forms turns out to be extremely simple and are closely connected with interpolatory polynomials of O. Kis [4] and of D. Jackson [3]. For other interesting results on Lacunary interpolation we refer to [5, 9, 11, 12].

2. Preliminaries. For the explicit representation of interpolatory polynomials we shall require known results of O. Kis [4].

$$u_{kn}(x) = \frac{1}{n}F(x - x_{kn}),\ v_{kn}(x) = \frac{2}{n^2}G(x - x_{kn}), \tag{2.1}$$

$k = 0,1,\cdots,n-1$, where

$$F(x) = 1 + \frac{2}{n}\sum_{j=1}^{n-1}\frac{(n-j)^2}{n-2j}\cos jx \tag{2.2}$$

$$= \frac{\sin\frac{nx}{2}}{\sin\frac{x}{2}} + \frac{1}{n}\sin\frac{nx}{2}\int_0^{x/2}\frac{n\sin t - \sin nt}{\sin^3 t}\cos t\,dt$$

$$G(x) = \frac{1}{2n} + \sum_{j=1}^{n-1}\frac{1}{n-2j}\cos jx - \frac{1}{2n}\cos nx \tag{2.3}$$

$$= \sin\frac{nx}{2}\int_0^{x/2}\sin nt \cot t\,dt.$$

These fundamental polynomials of (0, 2) interpolation satisfy the following conditions

$$F(x_k) = \begin{cases} n, & \text{for } k = 0 \quad F\,(x_k) = 0, \qquad 0 \leq |k| < n, \\ 0 & 0 < |k| < n, \end{cases} \tag{2.4}$$

(2.5) $$G(x_k) = 0 \quad 0 \leqq |k| < n,\ G''(x_k) = \begin{cases} \dfrac{n^2}{2} & \text{for } k = 0 \\ 0 & 0 < |k| < n. \end{cases}$$

The estimates of $F(x)$ and $G(x)$ are given by [4]

(2.6) $$|F(x)| \leqq 5n,\ |G(x) \leqq \pi,$$

and

(2.7) $$\sum_{k=0}^{n-1} |u_{kn}(\pi)| > \frac{n}{5},\ \sum_{k=0}^{n-1} |v_{kn}(\pi)| > \frac{1}{n}.$$

Finally we will require the fundamental polynomials of Hermite Fejer interpolation [13]

(2.8) $$H(x) = \frac{1}{n}\left[1 + \frac{2}{n}\sum_{j=1}^{n-1}(n-j)\cos jx\right] = \frac{1}{n^2}\left[\frac{\sin\frac{nx}{2}}{\sin\frac{x}{2}}\right]^2$$

which satisfy the conditions

(2.9) $$H(x_k) = \begin{cases} 1 \text{ for } k = 0,\ H'(x_k) = 0,\ 0 \leqq |k| < n \\ 0 \quad \text{for } 0 < |k| < n \end{cases}$$

and

(2.10) $$I(x) = \frac{2}{n^2}\sum_{j=1}^{n-1} \sin jx + \frac{\sin nx}{n^2} = \frac{2}{n^2}\sin^2\frac{nx}{2}\,\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}$$

which satisfy the conditions

(2.11) $$I(x_k) = 0 \quad 0 \leqq |k| < n-1,\ I'(x_k) = 1 \text{ for } k = 0$$
$$= 0\ \ 0 < |k| < n.$$

3. Statements of main theorems. We shall take n odd throughout this paper. The trigonometric polynomial $R_n(x)$ of order $2n$ satisfying the conditions

(3.1) $$R_n(x_{kn}) = f(x_{kn}),\ R_n'(x_{kn}) = a_{kn},\ R_n''(x_{kn}) = b_{kn},$$
$$R_n^{(IV)}(x_{kn}) = c_{kn},\ 0 \leqq |k| < n \text{ and of the form}$$

(3.2) $$d_0 + \sum_{i=1}^{2n-1}(d_i \cos ix + e_i \sin ix) + d_{2n}\cos 2nx$$

is given by

$$(3.3)\qquad R_n(x,f) = \sum_{i=0}^{n-1} f(x_{in})A(x-x_{in}) + \sum_{i=0}^{n-1} a_{in}B(x-x_{in}) + \sum_{i=0}^{n-1} b_{in}C(x-x_{in}) + \sum_{i=0}^{n-1} c_{in}D(x-x_{in}).$$

THEOREM 3.1. *For n odd we have the following representation of fundamental polynomials*

$$(3.4)\qquad D(x) = \frac{2}{3n^4}\sin^2\frac{nx}{2}G(x)$$

$$(3.5)\qquad C(x) = \frac{2}{n^3}\sin^2\frac{nx}{2}F(x) + n^2D(x)$$

$$(3.6)\qquad B(x) = \frac{1}{n}\sin nxH(x) - \frac{4}{3n^2}\sin^2\frac{nx}{2}H'(x)$$

$$(3.7)\qquad A(x) = H(x) - \frac{1}{2n}\sin nxH'(x) + \sin^2\frac{nx}{2}\left[\frac{1}{n}F(x) - H(x) + \frac{F''(x)}{n^3}\right].$$

THEOREM 3.2. *If $f(x)$ is a 2π periodic function satisfying the Zygmund condition λ.*

$$(3.8)\qquad |f(x+h) - 2f(x) + f(x-h)| = o(h)$$

with

$$(3.9)\qquad |a_{in}| = o(n),\ |b_{in}| = o(n),\ |c_{in}| = o(n^3),\ i = 0,1,\cdots,n-1\ .$$

Then the sequence $R_n(x,f)$ defined by 3.3) converges uniformly to $f(x)$ over the real line. Further Zygmund class can not be replaced by Lip α $0 < \alpha < 1$ even if all a_{in}, b_{in}, and c_{in} are zero.

4. The fundamental polynomials are determined from the conditions (3.2) and

$$(4.1)\qquad \begin{aligned} D^{(p)}(x_{kn}) &= 0,\ p = 0,1,2,\ 0 \leqq |k| < n,\\ D^{(IV)}(x_{kn}) &= 1 \text{ for } k = 0\\ &= 0\quad 0 < |k| < n,\end{aligned}$$

$$(4.2)\qquad \begin{aligned} C^{(p)}(x_{kn}) &= 0,\ p = 0,1,4,\ 0 \leqq |k| < n,\\ C''(x_{kn}) &= 1 \text{ for } k = 0\\ &= 0\quad 0 < |k| < n,\end{aligned}$$

$$(4.3)\qquad \begin{aligned} B^{(p)}(x_{kn}) &= 0,\ p=0,2,4,\ 0 \leqq |k| < n, \\ B'(x_{kn}) &= 1 \text{ for } k=0 \\ &= 0 \quad 0<|k|<n, \end{aligned}$$

and

$$(4.4)\qquad \begin{aligned} A(x_{kn}) &= 1 \text{ for } k=0 \qquad A^{(p)}(x_{kn}) = 0, \\ &= 0 \quad 0<|k|<n, \\ p &= 1,2,4,\ 0 \leqq |k| < n. \end{aligned}$$

The fundamental polynomials stated in Theorem 3.1 satisfy the above conditions. This can be easily verified on using (2.4), (2.5) and (2.9). Now the following lemma gives the estimates of the fundamental polynomials.

Lemma 4.1. *The following estimates are valid;*

$$(4.5)\qquad \sum_{i=0}^{n-1} |D(x-x_{in})| \leqq \frac{2\pi}{3n^3},\quad \sum_{i=0}^{n-1} |D(\pi-x_{in})| > \frac{1}{n^3}$$

$$(4.6)\qquad \sum_{i=0}^{n-1} |C(x-x_{in})| \leqq \frac{13}{n},\quad \sum_{i=0}^{n-1} |C(\pi-x_{in})| > \frac{c_3}{n}$$

$$(4.7)\qquad \sum_{i=0}^{n-1} |B(x-x_{in})| \leqq \frac{3}{n},\quad \sum_{i=0}^{n-1} \left|B\left(\frac{\pi}{2}-x_{kn}\right)\right| > \frac{c_1}{n}$$

and

$$(4.8)\qquad \sum_{i=0}^{n-1} |A(x-x_{in})| \leqq 11n \quad \sum_{i=0}^{n-1} |A(\pi-x_{in})| > c_2 n.$$

Proof. (4.5) follows from (3.4) and (2.6). Similarly the lower estimate of (4.5) follows from (3.4), (2.1) and second part of (2.7). Now (4.6) can be obtained from (2.6), (3.5) and first part of (4.5). From second part of (2.8) and simple differentiation we obtain

$$(4.9)\qquad \sin\frac{nx}{2} H'(x) = H(x)\left[n\cos\frac{nx}{2} - \cos\frac{x}{2}\,\frac{\sin\frac{nx}{2}}{\sin\frac{x}{2}}\right]$$

Combining (4.9), (2.10) and (3.6) we obtain

$$(4.10)\qquad B(x) = \frac{1}{3}\ H(x)\ \left[\frac{\sin nx}{n} + 2I(x)\right]$$

Obviously from (2.10) we have

$$|I(x)| \leqq \frac{2}{n}. \tag{4.11}$$

Therefore, on using positivity of $H(x)$ and the fact that

$$\sum_{i=0}^{n-1} H(x-x_{in}) = \sum_{i=0}^{n-1} |H(x-x_{in})| = 1 \tag{4.12}$$

we obtain

$$\sum_{i=0}^{n-1} |B(x-x_{in})| \leqq \frac{1}{3}\left[\frac{1}{n}+\frac{4}{n}\right] \leqq \frac{3}{n}.$$

In order to obtain lower estimate for $B(x)$ we first observe from (2.10) that

$$I\left(\frac{\pi}{2}\right) \leqq \frac{2}{n^2}. \tag{4.13}$$

Hence, on using (4.12) and (4.10) we obtain

$$\sum_{i=0}^{n-1}\left|B\left(\frac{\pi}{2}-x_{in}\right)\right| \geqq \frac{1}{3n}-\frac{4}{n^2} \geqq \frac{c_1}{n} \text{ for } n>4. \tag{4.14}$$

This proves the second part of (4.7).

To prove (4.8) we make use of (4.9) and (3.7) and we obtain

$$A(x) = \frac{1}{2n}H(x)\sin nx \cot\frac{x}{2} + \frac{(1-\cos nx)}{2n}\left[F(x)+\frac{F''(x)}{n^2}\right]. \tag{4.15}$$

Now using (2.6) and Bernstein inequality we obtain

$$|F(x)| \leqq 5n,\ |F''(x)| \leqq 5n^3. \tag{4.16}$$

Also

$$\left|\sin nx \cot\frac{x}{2}\right| \leqq 2n. \tag{4.17}$$

Therefore

$$\sum_{i=0}^{n-1} |A(x-x_{in})| \leqq 1+\frac{1}{n}[5n^2+5n^2] \leqq 11n.$$

In order to prove the lower estimate for $A(x-x_{in})$ we take $n=4p+1$ and observe that

$$F(\pi)+\frac{F''(\pi)}{n^2} = 1+\frac{4}{n^3}\sum_{j=1}^{2p}\alpha_j(-1)^j,\ \alpha_j = \frac{(2n-j)j^3}{n-2j}.$$

Moreover α_j is monotonic increasing for $j = 1,2\cdots,2p$. Using the fact that the series being alternating we obtain immediately

$$\left|\sum_{j=1}^{n-1} \alpha_j(-1)^j\right| \geqq \alpha_{2p} - \alpha_1 = c_2 n^4 \text{ with } c_2 > 0\,.$$

Now using (4.15) and the fact

$$A(\pi - x_{kn}) = A(\pi)$$

we obtain

$$\sum_{k=0}^{n-1} \left| A(\pi - x_{kn})\right| \geqq (4n-1)c_2 > c_3 n\,.$$

This proves second part of (4.8). The lower estimate of (4.6) is on the above lines so we omit the details.

LEMMA 4.2. *If $f(x)$ is continuous and 2π periodic and satisfies the Zygmund condition λ, then there exists a trigonometric polynomial $T_n(x)$ of order $n-1$ such that*

$$|f(x) - T_n(x)| = o\left(\frac{1}{n}\right) \tag{4.18}$$

$$|T_n'(x)| = 0(\log n), \quad |T_n^{(p)}(x)| = o(n^{p-1}), \quad p = 2,4\cdots \tag{4.19}$$

For the proof of this Lemma see O. Kis [4, page 271].

5. Proof of Theorem 3.2. By Lemma 4.2 there exists trigonometrical polynomial $T_n(x)$ which satisfies (4.18)-(4.19). But

$$f(x) - R_n(x) = f(x) - T_n(x) + T_n(x) - R_n(x)\,. \tag{5.1}$$

Since every trigonometric polynomial of order n can be uniquely expressed as

$$\begin{aligned} T_n(x) = &\sum_{k=0}^{n-1} T_n(x_{kn}) + \sum_{k=0}^{n-1} T_n'(x_{kn})B(x - x_{kn}) + \\ &+ \sum_{k=0}^{n-1} T_n''(x_{kn})C(x - x_{kn}) + \sum_{k=0}^{n-1} T_n^{(\mathrm{IV})}(x_{kn})D(x - x_{kn}). \end{aligned} \tag{5.2}$$

Therefore on using (3.3) we obtain

$$\begin{aligned} R_n(x) - T_n(x) = &\sum_{k=0}^{n-1} [T_n(x_{kn}) - f(x_{kn})]A(x - x_{kn}) \\ &+ \sum_{k=0}^{n-1} (T_n'(x_{kn}) - a_{kn})B(x - x_{kn}) + \sum_{k=0}^{n-1} (T_n''(x_{kn}) - b_{kn}) \\ &C(x - x_{kn}) + \sum_{k=0}^{n-1} (T_n^{(\mathrm{IV})}(x_{kn}) - c_{kn})D(x - x_{kn}) \\ = &\ J_1 + J_2 + J_3 + J_4 \text{ (say)}. \end{aligned} \tag{5.3}$$

From (4.18) and (4.8) we have

$$(J_1) \leqq 11 n\, o\left(\frac{1}{n}\right) = o(1)\,. \tag{5.4}$$

From (4.7) and (4.19) we have

$$\sum_{k=0}^{n-1} \left| T_n'(x_{kn}) B(x - x_{kn}) \right| \leqq \frac{3}{n} 0(\log n) = o(1), \tag{5.5}$$

and on using (4.7) and (3.9) we obtain

$$\sum_{k=0}^{n-1} \left| a_{kn} B(x - x_{kn}) \right| \leqq o(n) \frac{3}{n} = o(1)\,. \tag{5.6}$$

Therefore on using (5.5) and (5.6) we have

$$\left| J_2 \right| = o(1)\,. \tag{5.7}$$

On using (4.6) and (4.19) we obtain

$$\begin{aligned} \sum_{k=0}^{n-1} \left| T_n''(x_{kn}) C(x - x_{kn}) \right| &\leqq o(n) \sum_{k=0}^{n-1} \left| C(x - x_{kn}) \right| \\ &\leqq o(n) \frac{13}{n} = o(1), \end{aligned} \tag{5.8}$$

and, on using (46) and 3 9) we have

$$\sum_{k=0}^{n-1} \left| b_{kn} C(x - x_{kn}) \right| \leqq o(n) \frac{13}{n} = o(1) \tag{5 9}$$

on using (5 8) and 5 9) we obtain

$$\left| J_3 \right| = o(1) \tag{5.10}$$

on using (4.19) and 4 5) we have

$$\sum_{k=0}^{n-1} \left| T_n^{(\mathrm{IV})}(x_{kn}) D(x - x_{kn}) \right| \leqq o(n^3) \frac{2\pi}{3n^3} = o(1). \tag{5.11}$$

Lastly on using (4.5) and (39) we obtain

$$\sum_{k=0}^{n-1} \left| c_{kn} D(x - x_{kn}) \right| \leqq o(n^3) \frac{2\pi}{3n^3} = o(1)\,. \tag{5.12}$$

Therefore on using (5.11) and 517) we have

$$\left| J_4 \right| = o(1) \tag{5 13}$$

on using (5.4), (5.7), (5.10) and (5.13) we have

$$R_n(x) - T_n(x) = o(1)\,. \tag{5.14}$$

Thus (5.1), (4.18) and (5.14) we get

$$f(x) - R_n(x) = o(1) \ .$$

This proves the theorem. To show that we cannot replace Zygmund class by Lip $\alpha\, 0 < \alpha < 1$ we need the lower estimates of the fundamental polynomials given in Art 4 and we follow the similar proof as given in O. Kis [4] at the end of his paper.

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